7.5: Periodic and Piecewise Continuous Input Functions

Theorem 1. (Translation of the t-axis)
If $\mathcal{L}\{f(t)\}$ exists for $s > c$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

and

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

for $s > c + a$ where $u(t-a) = u_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$.

Example 1. Find $\mathcal{L}^{-1}\{\frac{e^{-as}}{s^3}\}$.

$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$. Therefore $\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s^3}\right\} = u(t-a)\frac{(t-a)^2}{2}$

$$= \begin{cases} 0, & t < a \\ \frac{1}{2}(t-a)^2, & t \geqslant a \end{cases}$$

Example 2. Find $\mathcal{L}\{g(t)\}$ if

$$g(t) = \begin{cases} 0 & \text{if } t < 3 \\ t^2 & \text{if } t \geq 3. \end{cases}$$

~~Let $f(t)=t^2$.~~

~~Then $g(t)=u(t-3)f(t-3)$~~

Let $f(t) = (t+3)^2$.

Then $g(t) = u(t-3)f(t-3)$.

Therefore $\mathcal{L}\{g(t)\} = e^{-3s} \cdot F(s)$

$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

Example 3. Find $\mathcal{L}\{f(t)\}$ if

$$f(t) = \begin{cases} \cos 2t & \text{if } 0 \leq t < 2\pi \\ 0 & \text{if } t \geq 2\pi. \end{cases}$$

$f(t) = [1 - u(t-2\pi)] \cdot \cos 2t$

$= \cos 2t - u(t-2\pi)\cos 2(t-2\pi)$.

So $F(s) = \frac{s}{s^2+4} - e^{-2\pi s} \cdot \frac{s}{(s^2+4)} = \frac{s(1-e^{-2\pi s})}{s^2+4}$.

Example 4. Consider the RLC circuit $R = 110\Omega$, $L = 1H$, $C = 0.001F$ and a battery supplying $E_0 = 90V$. Initially there is no current in the circuit and no charge on the capacitor. At time $t = 0$ the switch is closed and left closed for 1 second. At time $t = 1$ it is opened and left open thereafter. Find the resulting current in the circuit if the equation is given by

$$\frac{di}{dt} + 110i + 1000q = e(t).$$

Recall $i = \frac{dq}{dt}$; i.e. $q = \int_0^t i(\tau)d\tau$. Also $e(t) = 90[1-u(t-1)]$.

Using the fact that $\mathcal{L}\{\int_0^t i(\tau)d\tau\} = \frac{I(s)}{s}$, we get (after transform)

$$sI(s) + 110I(s) + 1000\frac{I(s)}{s} = \frac{90}{s}(1-e^{-s})$$

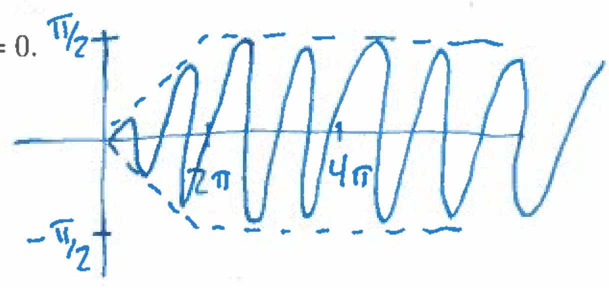
Then $I(s) = \dfrac{90(1-e^{-s})}{s^2+110s+1000} = \dfrac{1}{s+10} - \dfrac{1}{s+100} - e^{-s}\left(\dfrac{1}{s+10} - \dfrac{1}{s+100}\right)$.

Thus $i(t) = \mathcal{L}^{-1}\{I(s)\} = e^{-10t} - e^{-100t} - u(t-1)\left[e^{-10(t-1)} - e^{-100(t-1)}\right]$

$$= \begin{cases} e^{-10t} - e^{-100t}, & t<1 \\ e^{-10t} - e^{-10(t-1)} - e^{-100t} + e^{-100(t-1)}, & t \geq 1 \end{cases}$$

Exercise 1. A mass that weighs 32 lb is attached to the free end of a long, light spring that is stretched 1ft by a force of 4 lb. The mass is initially at rest in its equilibrium position. Beginning at time $t = 0$ (seconds), an external force $f(t) = \cos 2t$ is applied to the mass, but at time $t = 2\pi$ this force is turned off (abruptly discontinued) and the mass is allowed to continue its motion unimpeded. Find the resulting position function $x(t)$ of the mass is the resulting differential equation is

$$x'' + 4x = f(t); \quad x(0) = x'(0) = 0.$$



$f(t) = [1-u(t-2\pi)] \cdot \cos 2t$.

Thus $(s^2+4)X(s) = \dfrac{s(1-e^{-2\pi s})}{s^2+4}$

and $X(s) = \dfrac{s(1-e^{-2\pi s})}{(s^2+4)^2}$.

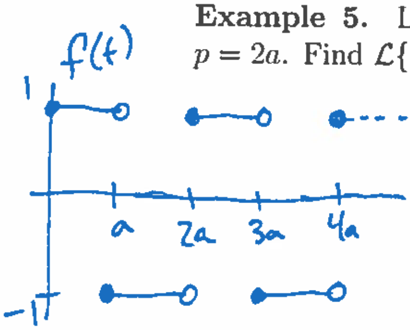
Therefore

$$x(t) = \frac{1}{4}[t - u(t-2\pi)(t-2\pi)]\sin 2t = \begin{cases} \frac{1}{4}t\sin 2t, & t<2\pi \\ \frac{1}{2}\pi \sin 2t, & t \geq 2\pi \end{cases}$$

Theorem 2. (Transforms of Periodic Functions)
Let $f(t)$ be periodic with period p and piecewise continuous for $t \geq 0$. Then the transform $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > 0$ and is given by

$$F(s) = \frac{1}{1-e^{-ps}} \int_0^p e^{-} st f(t) dt.$$

Example 5. Let $f(t) = (-1)^{\lceil t/a \rceil}$ be the square-wave function of period $p = 2a$. Find $\mathcal{L}\{f(t)\}$.



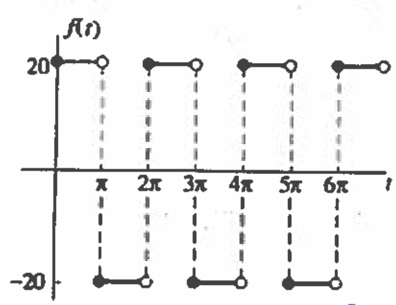
$$F(s) = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t) dt$$

$$= \frac{1}{1-e^{-2as}} \left(\int_0^a e^{-st} dt + \int_a^{2a} e^{-st} \cdot (-1) dt \right)$$

$$= \frac{(1-e^{-as})^2}{s(1-e^{-2as})} = \frac{1-e^{-as}}{s(1+e^{-as})} = \frac{1}{s} \tanh \frac{as}{2}.$$

Example 6. Consider a mass-spring-dashpot system with $m = 1$, $c = 4$, and $k = 20$ in appropriate units. Suppose that the system is initially at rest at equilibrium and that the mass is acted on beginning at time $t = 0$ by the external force $f(t)$ which is the square wave function with amplitude 20 and period 2π. Find the position function $x(t)$ is the associated differential equation is

$$x'' + 4x' + 20x = f(t); \quad x(0) = x'(0) = 0.$$



$$s^2X(s) + 4sX(s) + 20X(s) = F(s) = \frac{20}{s} \cdot \frac{1-e^{-\pi s}}{1+e^{-\pi s}} = \frac{20}{s} + \frac{40}{s} \sum_{n=1}^{\infty} (-1)^n e^{-n\pi s}$$

So $$X(s) = \frac{F(s)}{s^2+4s+20} = \frac{20}{s[(s+2)^2+16]} + 2 \sum_{n=1}^{\infty} (-1)^n \frac{20e^{-n\pi s}}{s[(s+2)^2+16]}.$$

Noticing that $\mathcal{L}^{-1}\left\{\frac{20}{(s+2)^2+16}\right\} = 5e^{-2t} \sin 4t$

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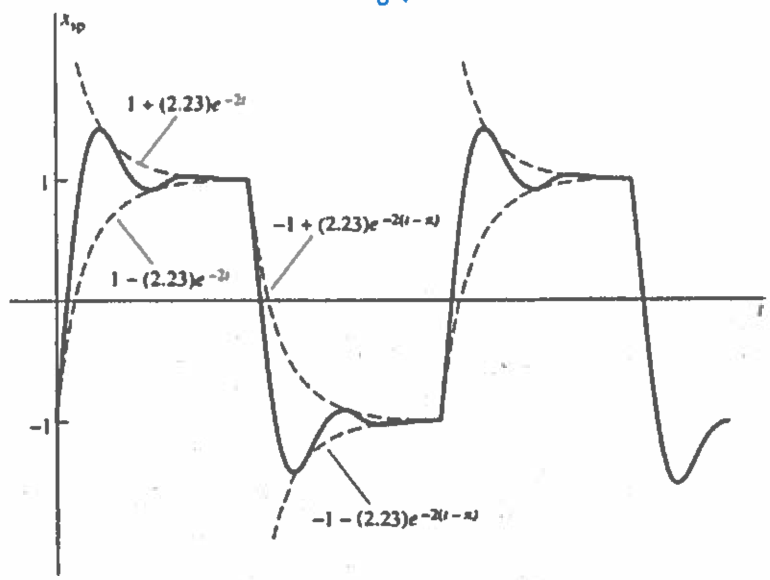
Particular Solution.



We get $\mathcal{L}^{-1}\left\{\dfrac{20}{s[(s+2)^2+16]}\right\} = \int_0^t 5e^{-2\tau}\sin 4\tau\, d\tau = 1 - e^{-2t}\left(\cos 4t + \frac{1}{2}\sin 4t\right)$

Therefore

$$x(t) = \left(1 - e^{-2t}\left(\cos 4t + \tfrac{1}{2}\sin 4t\right)\right) + 2\sum_{n=1}^{\infty} u(t-n\pi)\cdot(-1)^n\left(1 - e^{-2(t-n\pi)}\left(\cos 4(t-n\pi) + \tfrac{1}{2}\sin 4(t-n\pi)\right)\right)$$

$$= \left(1 - e^{-2t}\left(\cos 4t + \tfrac{1}{2}\sin 4t\right)\right) + 2\sum_{n=1}^{\infty} u(t-n\pi)\cdot(-1)^n\left(1 - e^{n\pi}e^{-2t}\left(\cos 4t + \tfrac{1}{2}\sin 4t\right)\right)$$

So that for $n\pi < t < (n+1)\pi$,

$$x(t) = \frac{e^{2\pi}-1}{e^{2\pi}+1}e^{-2t}\left(\cos 4t + \tfrac{1}{2}\sin 4t\right) + (-1)^n - \frac{2\cdot(-1)^n e^{2\pi}}{e^{2\pi}+1}e^{-2(t-n\pi)}\left(\cos 4t + \tfrac{1}{2}\sin 4t\right)$$

$$\simeq \underbrace{(1.1139)e^{-2t}\cos(4t-0.46)}_{\text{characteristic}} + \underbrace{(-1)^n\left[1-(2.2314)e^{-2t}\left(\cos(4t-0.4636)\right)\right]}_{\text{particular}}$$

Homework. 1-29, 33-37 (odd)